MATH 303 — MEASURE THEORY
SUPPLEMENTARY MATERIALS ON
METRIC SPACES AND TOPOLOGICAL SPACES
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Contents

1. Metric spaces 1
1.1. Examples of metric spaces 1
1.2. Limits and continuity 2
1.3. Open and closed sets 3
1.4. Properties of metric spaces 3
2. Topological spaces 6
2.1. Subsets of topological spaces 7
2.2. Bases 7
2.3. Limits and continuity 7
2.4. Properties of topological spaces 8

This document is a very brief overview of concepts from point set topology. It is not expected for this course that you have a complete mastery of the subject, but we will use various concepts covered in this document throughout the course. You may which to return to this document when relevant as the semester goes along.

You should be aware that the document is light on exposition and heavy on definitions. I have tried to include interesting and illustrative examples and exercises to help with understanding the definitions. However, if you are struggling with any of the concepts, you may wish to consult a longer reference on metric spaces or topology.

1. Metric spaces

Metric spaces are an abstract mathematical object capturing the essential features of the notion of "distance."

Definition 1.1

A *metric* (or *distance*) on a set X is a function $d : X \times X \to [0, \infty)$ satisfying the following properties for all $x, y, z \in X$:

- positive definite: $d(x, y) = 0 \iff x = y$;
- symmetric: $d(x, y) = d(y, x)$; and
- triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

If d is a metric on X, we say that the pair (X, d) is a *metric space*.

1.1. Examples of metric spaces.

Example 1.2: Real numbers as a metric space

The real numbers are a metric space with the metric $d(x, y) = |x - y|$.

Example 1.3: Metrics on Euclidean spaces

Let $k \in \mathbb{N}$. The Euclidean space $\mathbb{R}^k$ can be equipped with several different metrics. These include:

- Euclidean distance: $d_2(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{j=1}^{k} (x_j - y_j)^2}$
- Manhattan or taxicab distance: $d_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{k} |x_j - y_j|$
- Chebyshev distance: $d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_{1 \le j \le k} |x_j - y_j|$

Example 1.4: Discrete metric

Every set can be made into a metric space with the discrete metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

Example 1.5: Graph distance

Every connected graph can be viewed as a metric space by defining, for pairs of vertices u, v,

$$d(u, v) = \# \text{ of edges in the shortest path from } u \text{ to } v.$$

Exercise 1.1

Check that the spaces in the above examples are metric spaces.

1.2. **Limits and continuity.** Since metric spaces come with a notion of points being "close" to one another, they are a natural setting for dealing with familiar notions from analysis such as limits, continuity, etc.

Definition 1.6

Let (X, d) be a metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and $x \in X$. We say that $(x_n)_{n \in \mathbb{N}}$ *converges* to x, written $\lim_{n \to \infty} x_n = x$, if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \ge N \implies d(x_n, x) < \varepsilon.$$

Definition 1.7

Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$.

- f is *continuous at a point $x \in X$* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x' \in X$ and $d(x', x) < \delta$, then $\rho(f(x'), f(x)) < \varepsilon$.
- f is *continuous* if it is continuous at every point $x \in X$.
- f is *uniformly continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, then $\rho(f(x_1), f(x_2)) < \varepsilon$.

REMARK. Note that this agrees with the usual definition of limits and continuity in the real numbers using the metric from Example 1.2.

EXERCISE 1.2

Show that every uniformly continuous function is continuous. Give an example of a function that is continuous but not uniformly continuous.

EXERCISE 1.3

Let (X, d) and (Y, ρ) be metric spaces, let $f : X \to Y$, and let $x \in X$. Show that f is continuous at x if and only if for every sequence $(x_n)_{n\in\mathbb{N}}$, if $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} f(x_n) = f(x)$.

1.3. Open and closed sets.

DEFINITION 1.8

Let (X, d) be a metric space.

- The *open ball* of radius $r > 0$ centered at a point $x \in X$ is the set $B(x, r) = \{y \in X : d(x, y) < r\}$.
- The *closed ball* of radius $r > 0$ centered at a point $x \in X$ is the set $B[x, r] = \{y \in X : d(x, y) \leq r\}$.
- A set $U \subseteq X$ is *open* if for every $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$.
- A set $C \subseteq X$ is *closed* if every point approximable by elements of C belongs to C. That is, if $x \in X$ and $B(x, r) \cap C \neq \emptyset$ for every $r > 0$, then $x \in C$.
- A set $D \subseteq X$ is *dense* if every point can be approximated arbitrarily well by elements of D. That is, for every $x \in X$ and every $\varepsilon > 0$, there exists $y \in D$ such that $d(x, y) < \varepsilon$.
- The *closure* of a set $E \subseteq X$, denoted by $\overline{E}$, is the smallest closed set containing E.
- The *interior* of a set $E \subseteq X$, denoted by E°, is the largest open set contained in E.

EXERCISE 1.4

Prove the following properties of open and closed sets:

(a) For every $x \in X$ and $r > 0$, the open ball $B(x, r)$ is open.
(b) For every $x \in X$ and $r > 0$, the closed ball $B[x, r]$ is closed.
(c) There exists a metric space (X, d), a point $x \in X$, and a radius $r > 0$ such that $\overline{B(x, r)} \neq B[x, r]$.
(d) A set is open if and only if its complement is closed.
(e) An arbitrary intersection of closed sets is closed.
(f) An arbitrary union of open sets is open.
(g) The closure and interior of a set are well-defined. (Hint: use (b) and (c).)
(h) A set D is dense in X if and only if $\overline{D} = X$.

1.4. **Properties of metric spaces.** We now discuss different properties that metric spaces may possess. To enable this discussion, we need another definition.

Definition 1.9

Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ is a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \implies d(x_n, x_m) < \varepsilon.$$

Exercise 1.5

Show that every convergent sequence in a metric space is Cauchy.

In a Cauchy sequence, the terms cluster together and appear to *want* to converge. However, convergence is not guaranteed.

Example 1.10

Consider $\mathbb{Q}$ as a metric space with metric $d(x, y) = |x - y|$. Define a sequence $(x_n)_{n\in\mathbb{N}}$ recursively as follows:

$$x_0 = 2$$
$$x_n = \frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}.$$

Since we start with a rational value for x_0, it is clear from the recurrence relation that x_n is rational for every $n \in \mathbb{N}$. By the arithmetic mean-geometric mean inequality,

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right) > \sqrt{x_{n-1} \cdot \frac{2}{x_{n-1}}} = \sqrt{2}.$$

It follows that $(x_n)_{n\in\mathbb{N}}$ is a decreasing sequence, since

$$x_n = x_{n-1} + \frac{1}{x_{n-1}} \underbrace{\left(-\frac{x_{n-1}^2}{2} + 1\right)}_{<0} < x_{n-1}.$$

Thus, $(x_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence in the interval $\mathbb{Q} \cap (\sqrt{2}, 2]$. One can check that bounded monotone sequences are always Cauchy in $\mathbb{Q}$.

However, the sequence $(x_n)_{n\in\mathbb{N}}$ does not converge in $\mathbb{Q}$. To see this, suppose for contradiction that $x = \lim_{n\to\infty} x_n$. Then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{x_{n-1}}{2} + \frac{1}{x_{n-1}}\right) = \frac{x}{2} + \frac{1}{x}.$$

Solving for x gives $x = \sqrt{2}$, but $\sqrt{2}$ is an irrational number, so $(x_n)_{n\in\mathbb{N}}$ does not converge in $\mathbb{Q}$.

The failure of convergence in Example 1.10 comes from the fact that the rational numbers have "holes" that are filled in with irrationals in the reals. Spaces without such "holes" are called complete.

Definition 1.11

A metric space is *complete* if every Cauchy sequence converges.

Theorem 1.12

Given a metric space (X, d), there exists a space $(\widetilde{X}, \widetilde{d})$ such that

- $(\widetilde{X}, \widetilde{d})$ is complete;
- there is an isometry[a] $\iota : X \to \widetilde{X}$; and
- $\iota(X)$ is dense in $\widetilde{X}$.

[a] An isometry is a distance-preserving map. Saying that ι is an isometry means $\widetilde{d}(\iota(x_1), \iota(x_2)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$.

Remark. The space $(\widetilde{X}, \widetilde{d})$ is called the *completion* of (X, d).

Example 1.13

The real numbers are the completion of the rational numbers.

Proof of Theorem 1.12. (Sketch) Let Y be the space of Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ in X. Define an equivalence relation on Y by

$$(x_n)_{n\in\mathbb{N}} \sim (x'_n)_{n\in\mathbb{N}} \iff \lim_{n\to\infty} d(x_n, x'_n) = 0.$$

We then let $\widetilde{X} = Y/\sim$ be the space of equivalence class of sequences with the metric

$$\widetilde{d}\left([(x_n)_{n\in\mathbb{N}}], [(x'_n)_{n\in\mathbb{N}}]\right) = \lim_{n\to\infty} d(x_n, x'_n).$$

The map $\iota : X \to \widetilde{X}$ is defined by $\iota(x) = (x, x, x, \dots)$ for $x \in X$. Given a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$, the corresponding sequence $(\iota(x_n))_{n\in\mathbb{N}}$ in $\widetilde{X}$ now converges to the point $[(x_n)_{n\in\mathbb{N}}] \in \widetilde{X}$. □

One of the main theorems concerning convergence of sequences in the real numbers is the Bolzano–Weierstrass theorem, which says that every bounded sequence of real numbers has a convergent subsequence. This motivates the following definition.

Definition 1.14

Let (X, d) be a metric space. A set $K \subseteq X$ is *sequentially compact* if every sequence in K has a convergent subsequence in K. That is, for every $(x_n)_{n\in\mathbb{N}}$ in K, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{n_k} \in K$.

It turns out that for metric spaces, a set being sequentially compact is equivalent to another notion of compactness given in the next definition.

Definition 1.15

Let (X, d) be a metric space. A set $K \subseteq X$ is *compact* if every open cover of K has a finite subcover. That is, if $(U_i)_{i\in I}$ is a family of open sets and $K \subseteq \bigcup_{i\in I} U_i$, then there exists a finite set $\{i_1, \dots, i_N\} \subseteq I$ such that $K \subseteq \bigcup_{j=1}^{N} U_{i_j}$.

Theorem 1.16

Let (X, d) be a metric space. The following are equivalent:

(i) X is sequentially compact;
(ii) X is compact;
(iii) X is complete and totally bounded (i.e., for every $\varepsilon > 0$, there is a finite set $\{x_1, \ldots, x_N\} \subseteq X$ such that $X = \bigcup_{j=1}^{N} B(x_j, \varepsilon)$).

Exercise 1.6

Prove Theorem 1.16.

Compact subsets of the real line are characterized by the Heine–Borel theorem.

Theorem 1.17: Heine–Borel

A subset $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

2. Topological spaces

Topological spaces involve another level of abstraction, dispensing with the notion of distance but retaining a notion of open sets.

Definition 2.1

A *topology* on a set X is a collection of subsets of X, $\tau \subseteq \mathscr{P}(X)$, such that

- $\emptyset, X \in \tau$;
- if $\{U_i\}_{i \in I} \subseteq \tau$, then $\bigcup_{i \in I} U_i \in \tau$;
- if $U, V \in \tau$, then $U \cap V \in \tau$.

Elements of τ are called *open sets*, and the pair (X, τ) is called a *topological space*.

Example 2.2: Discrete and anti-discrete topology

Let X be a set. The *discrete topology* on X is $\tau_{\text{disc}} = \mathscr{P}(X)$, and the *anti-discrete topology* on X is $\tau_{\text{anti-disc}} = \{\emptyset, X\}$.

Example 2.3: Metric spaces

Let (X, d) be a metric space. The open sets (as defined in Definition 1.8) form a topology on X.

Exercise 2.1

Check that the above examples satisfy the definition of a topology.

Remark. When discussing topological spaces, it is typical to omit explicit reference to τ and simply refer to its elements as open sets. In line with this standard mathematical practice, for the remainder of this document we will write, "X is a topological space," with the understanding that this entails an implicit collection of open sets.

2.1. **Subsets of topological spaces.** Motivated by the corresponding definitions for metric spaces, we introduce terminology for special classes of subsets of a topological space.

Definition 2.4

Let X be a topological space.

- A set is *closed* if its complement is open.
- The *closure* of a set E is the smallest closed set containing E.
- The *interior* of E is the largest open set contained in E.
- A set D is *dense* in X if every nonempty open set in X intersects D.
- A set $K \subseteq X$ is *compact* if every open cover of K has a finite subcover.

2.2. **Bases.** It is often impractical to specify a topology by listing every possible open set. One instead frequently defines a topology in terms of a basis (or base).

Definition 2.5

A *basis* (or *base*) for a topological space X is a family $\mathcal{B}$ of open sets such that

- for every point $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$,
- for every $B_1, B_2 \in \mathcal{B}$ and every point $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

One should think of the basis axioms as corresponding to the assumption that X is open and that the intersection of two open sets is again an open set. The condition that arbitrary unions of open sets are open does not appear anywhere in the definition of a basis; this is because the topology generated by the basis consists precisely of arbitrary unions of basis elements.

Example 2.6

In a metric space, the family of open balls forms a basis.

Example 2.7: Furstenberg topology on the integers

For $a, b \in \mathbb{Z}$ with $a \neq 0$, let $S(a,b) = \{an + b : n \in \mathbb{Z}\} = a\mathbb{Z} + b$. Then $(S(a,b))_{a,b\in\mathbb{Z},a\neq 0}$ is the basis for a topology on $\mathbb{Z}$.

Exercise 2.2

Show that each basis element $S(a,b)$ is also closed in the Furstenberg topology. Use the fundamental theorem of arithmetic to find an expression for the set $\mathbb{Z}\backslash\{-1,+1\}$ and conclude that there are infinitely many prime numbers.

2.3. **Limits and continuity.** We now extend notions of convergence and continuous functions to the setting of topological spaces.

Definition 2.8

A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X *converges* to $x \in X$ if for every open neighborhood $U \ni x$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies x_n \in U.$$

Definition 2.9

Let X and Y be topological spaces. A function $f : X \to Y$ is *continuous* if for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

Exercise 2.3

Check that these definitions agree with the corresponding definitions for metric spaces.

Exercise 2.4

Suggest a definition for what it means for f to be *continuous at a point* $x \in X$. The definition should generalize the definition for metric spaces.

2.4. **Properties of topological spaces.** There are many properties that topological spaces may or may not have. We give a list of various properties that will play a role in this course.

Definition 2.10

Let X be a topological space. We say that X is

- *second countable* if there is a countable basis for the topology
- *separable* if there is a countable dense subset
- *locally compact* if for every $x \in X$, there is an open set $U \ni X$ such that $\overline{U}$ is compact
- *Hausdorff* if for every pair of distinct points $x, y \in X$, $x \neq y$, there are open sets $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$
- *metrizable* if there is a metric inducing the topology
- *completely metrizable* if it is metrizable and there is a compatible metric for which X is complete
- *Polish* if it is separable and completely metrizable

Exercise 2.5

Give an example of a non-metrizable topological space.

Exercise 2.6

Show that every metric space is Hausdorff.

Exercise 2.7

(a) Show that every second countable space is separable.
(b) Show that if X is a separable metric space, then it is second countable.
(c) Give an example of a separable topological space that is not second countable.